Phase Retrieval with Gabor Frames: Stability and Reconstruction Algorithms

Palina Salanevich

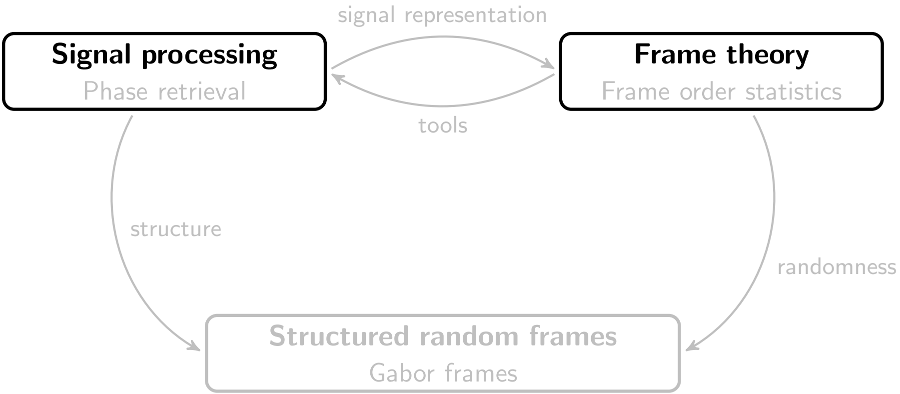
Email: p.salanevich@uu.nl

Utrecht University

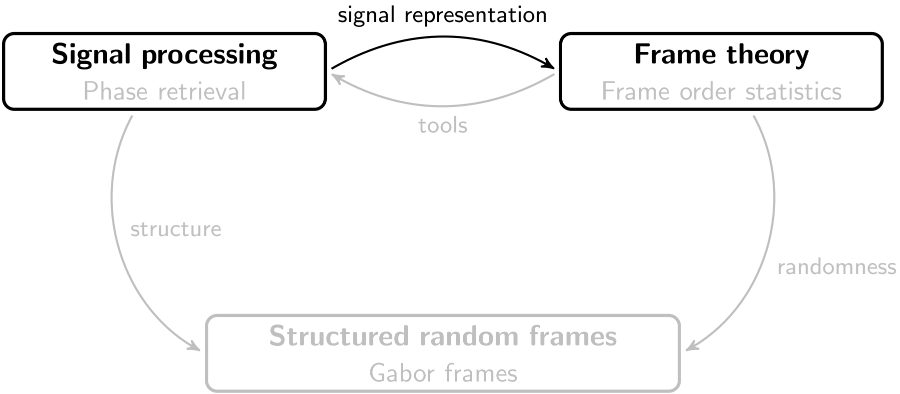
June 27, 2022

CWI Inverse Problems Seminar

Signal processing and frames



Signal processing and frames



Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

- We identify a frame $\Phi = \{\varphi_j\}_{j=1}^N$ with its synthesis matrix $\Phi \in \mathbb{C}^{M \times N}$ that has vectors φ_j as its columns. Its adjoint Φ^* is the analysis matrix of Φ.

Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

- We identify a frame $\Phi = \{\varphi_j\}_{j=1}^N$ with its synthesis matrix $\Phi \in \mathbb{C}^{M \times N}$ that has vectors φ_j as its columns. Its adjoint Φ^* is the analysis matrix of Φ.
- The vector $\Phi^* x = (\langle x, \varphi_j \rangle)_{j=1}^N$ is called the vector of frame coefficients of x.

Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

- We identify a frame $\Phi = \{\varphi_j\}_{j=1}^N$ with its synthesis matrix $\Phi \in \mathbb{C}^{M \times N}$ that has vectors φ_j as its columns. Its adjoint Φ^* is the analysis matrix of Φ.
- The vector $\Phi^* x = (\langle x, \varphi_j \rangle)_{j=1}^N$ is called the vector of frame coefficients of x.
- The signal x can be reconstructed from the vector of its frame coefficients using a dual frame $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}_{j=1}^N$ as $x = \sum_{j=1}^N \langle x, \varphi_j \rangle \widetilde{\varphi}_j = \widetilde{\Phi}\Phi^* x$.

Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

- We identify a frame $\Phi = \{\varphi_j\}_{j=1}^N$ with its synthesis matrix $\Phi \in \mathbb{C}^{M \times N}$ that has vectors φ_j as its columns. Its adjoint Φ^* is the analysis matrix of Φ.
- The vector $\Phi^* x = (\langle x, \varphi_j \rangle)_{j=1}^N$ is called the vector of frame coefficients of x.
- The signal x can be reconstructed from the vector of its frame coefficients using a dual frame $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}_{j=1}^N$ as $x = \sum_{j=1}^N \langle x, \varphi_j \rangle \widetilde{\varphi}_j = \widetilde{\Phi}\Phi^* x$.
- The standard dual frame $\widetilde{\Phi} = (\Phi\Phi^*)^{-1}\Phi$.

Example: music processing

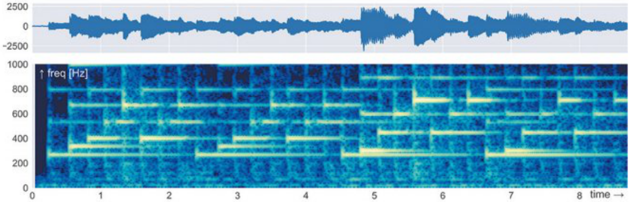


Figure: Time-frequency representation of a music piece: the measurement frame Φ in this case is a Gabor frame.

Example: music processing

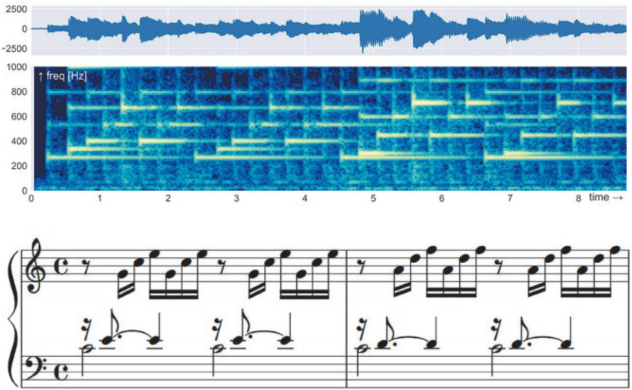


Figure: Time-frequency representation of a music piece: the measurement frame Φ in this case is a Gabor frame. There is a similarity between this time-frequency representation and musical staff notation.

Example: diffraction imaging

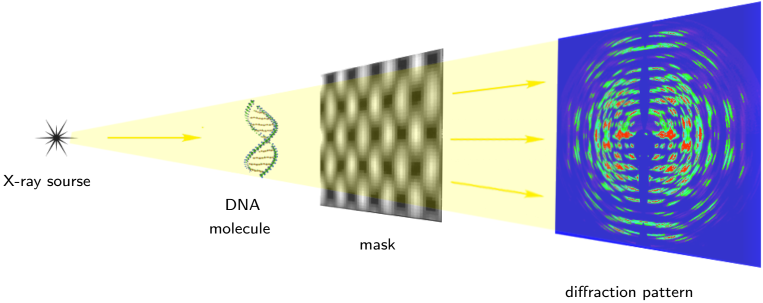
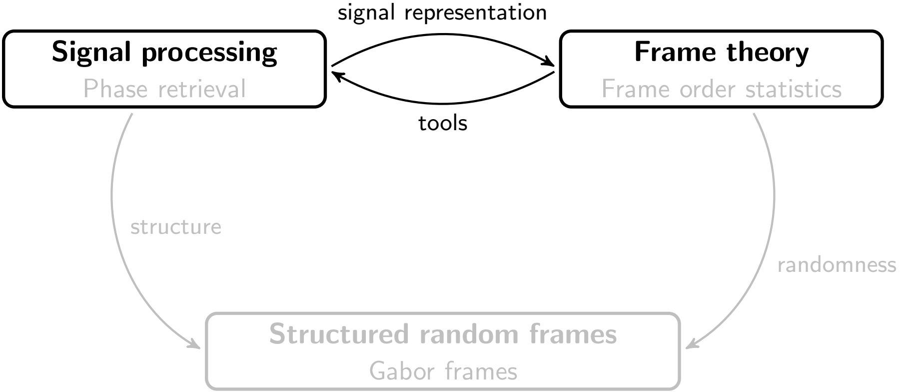
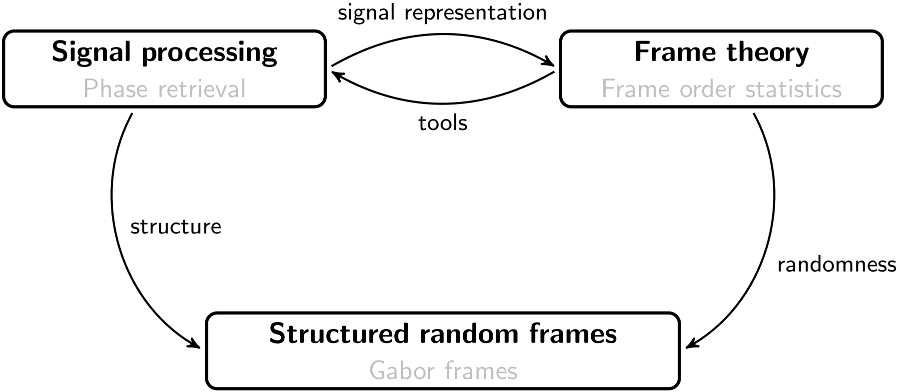


Figure: A typical setup for structured illuminations in *diffraction imaging* using a *phase mask*. The measurement map in this case is given by $\mathcal{A} : x \mapsto \{|\mathcal{F}(x \odot Q)(\ell)|^2\}_{\ell \in \Omega}$, where $\odot$ denotes pointwise multiplication and Q is a mask placed on the way of the scattered waves. The measurement frame $\Phi = \{\bar{Q} \odot e^{2\pi i \ell (\cdot)}\}_{\ell \in \mathbb{Z}_M}$.

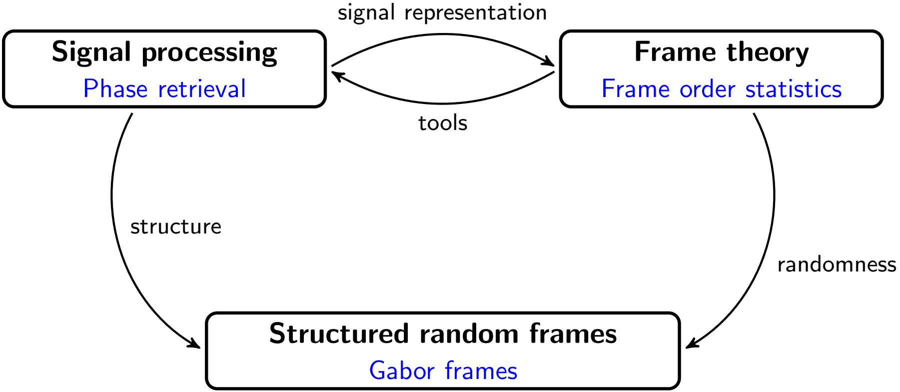
Signal processing and frames: structure and randomness



Signal processing and frames: structure and randomness



Signal processing and frames: structure and randomness



Phase retrieval: problem statement

For a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$, define the measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$, $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$. For a given vector of measurements $b \in \mathbb{R}^N$, we address the following non-convex inverse problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \mathcal{A}_\Phi(x) = b. \end{array}$$

Phase retrieval: problem statement

For a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$, define the measurement map $\mathcal{A}_\Phi : \mathbb{C}^M/_\sim \to \mathbb{R}^N,\ \mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$, where $x \sim e^{i\theta}x$ for any $\theta \in [0, 2\pi)$.
For a given vector of measurements $b \in \mathbb{R}^N$, we address the following non-convex inverse problem

find	x
subject to	$\mathcal{A}_\Phi(x) = b.$

Phase retrieval: problem statement

For a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$, define the measurement map $\mathcal{A}_\Phi : \mathbb{C}^M/_\sim \to \mathbb{R}^N$, $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$, where $x \sim e^{i\theta}x$ for any $\theta \in [0, 2\pi)$.
For a given vector of measurements $b \in \mathbb{R}^N$, we address the following non-convex inverse problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \mathcal{A}_\Phi(x) = b. \end{array}$$

Question

1. *For which Φ is $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$ injective and stable?*
2. *For a given Φ, how to efficiently recover x from $\mathcal{A}_\Phi(x)$ (algorithms)?*

Phase retrieval: problem statement

For a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$, define the measurement map $\mathcal{A}_\Phi : \mathbb{C}^M/_\sim \to \mathbb{R}^N$, $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$, where $x \sim e^{i\theta}x$ for any $\theta \in [0, 2\pi)$. For a given vector of measurements $b \in \mathbb{R}^N$, we address the following non-convex inverse problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \mathcal{A}_\Phi(x) = b. \end{array}$$

Question

1. *For which Φ is $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$ injective and stable?*
2. *For a given Φ, how to efficiently recover x from $\mathcal{A}_\Phi(x)$ (algorithms)?*

- Mostly studied for Gaussian frames: with $\varphi_j(m) \sim i.i.d.\ \mathcal{CN}(0, 1/n)$.
- Very little is known for structured, application relevant frames, such as Gabor frames.

Gabor frames and phase retrieval applications

Definition (Gabor frames)

For a window $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the Gabor frame is given by $(g, \Lambda) = \{\pi(k, \ell)g\}_{(k,\ell)\in\Lambda}$, where

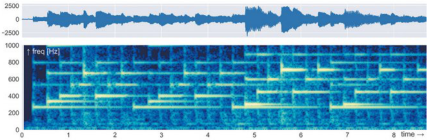
1. $\pi(k, \ell) = M_\ell T_k$ is a time-frequency shift operator;
2. $T_k x = (x(m-k))_{m\in\mathbb{Z}_M}$ is translation operator;
3. $M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m\in\mathbb{Z}_M}$ is modulation operator.

Gabor frames and phase retrieval applications

Definition (Gabor frames)

For a window $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the Gabor frame is given by $(g, \Lambda) = \{\pi(k, \ell)g\}_{(k,\ell)\in\Lambda}$, where

1. $\pi(k, \ell) = M_\ell T_k$ is a time-frequency shift operator;
2. $T_k x = (x(m-k))_{m\in\mathbb{Z}_M}$ is translation operator;
3. $M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m\in\mathbb{Z}_M}$ is modulation operator.



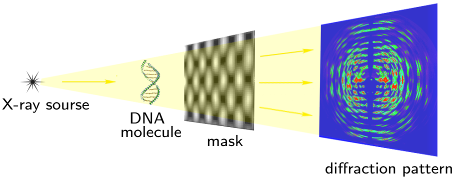
Speech recognition and music separation: $\mathcal{A}(x) = \{|\langle x, \pi(k, \ell)g\rangle|^2\}_{(k,\ell)\in\mathbb{Z}_M\times\mathbb{Z}_M}$, where g is a mask (spectrograms).
The measurement frame $\Phi = (g, \mathbb{Z}_M \times \mathbb{Z}_M)$ is a Gabor frame.

Gabor frames and phase retrieval applications

Definition (Gabor frames)

For a window $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the Gabor frame is given by $(g, \Lambda) = \{\pi(k,\ell)g\}_{(k,\ell)\in\Lambda}$, where

1. $\pi(k,\ell) = M_\ell T_k$ is a time-frequency shift operator;
2. $T_k x = (x(m-k))_{m\in\mathbb{Z}_M}$ is translation operator;
3. $M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m\in\mathbb{Z}_M}$ is modulation operator.



Diffraction imaging:
$\mathcal{A}(x) = \{|\mathcal{F}(x \odot g)(\ell)|^2\}_{\ell\in\Omega} = \{|\langle x, M_\ell \bar{g}\rangle|^2\}_{\ell\in\Omega}$, where $\odot$ denotes pointwise multiplication and g is a mask. If several shifts of the same mask are used, the measurement frame $\Phi = \{M_\ell T_k \bar{g}\}_{\ell\in\mathbb{Z}_M, k\in F} = (\bar{g}, F \times \mathbb{Z}_M)$ is a Gabor frame.

Injectivity of phase retrieval

Question

For which frames Φ *is the associated phaseless measurement map* $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$ *injective and stable?*

Injectivity of phase retrieval

Question

For which frames Φ is the associated phaseless measurement map $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$ injective and stable?

- (Heinosaari, Mazzarella and Wolf) If $|\Phi| < (4 + o(1))M$, then $\mathcal{A}_\Phi$ is not injective.

Teiko Heinosaari, Luca Mazzarella and Michael M. Wolf
Quantum tomography under prior information, *Communications in Mathematical Physics* 318(2), 355–374, 2013.

Injectivity of phase retrieval

Question

For which frames Φ is the associated phaseless measurement map $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$ injective and stable?

- (Heinosaari, Mazzarella and Wolf) If $|\Phi| < (4 + o(1))M$, then $\mathcal{A}_\Phi$ is not injective.
- (Conca, Edidin, Hering, and Vinzant and, independently, Király and Ehler) Map $\mathcal{A}_\Phi$ is injective for a generic Φ with $|\Phi| \geq 4M - 4$.

Teiko Heinosaari, Luca Mazzarella and Michael M. Wolf
Quantum tomography under prior information, *Communications in Mathematical Physics* 318(2), 355–374, 2013.

Aldo Conca, Dan Edidin, Milena Hering and Cynthia Vinzant
An algebraic characterization of injectivity in phase retrieval, *Applied and Computational Harmonic Analysis* 38(2), 346–356, 2015.

Franz J. Király and Martin Ehler
The algebraic approach to phase retrieval and explicit inversion at the identifiability threshold, *arXiv:1402.4053*, 2014.

Injectivity of phase retrieval

Question

For which frames Φ is the associated phaseless measurement map $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$ injective and stable?

- (Heinosaari, Mazzarella and Wolf) If $|\Phi| < (4 + o(1))M$, then $\mathcal{A}_\Phi$ is not injective.
- (Conca, Edidin, Hering, and Vinzant and, independently, Király and Ehler) Map $\mathcal{A}_\Phi$ is injective for a generic Φ with $|\Phi| \geq 4M - 4$.

Teiko Heinosaari, Luca Mazzarella and Michael M. Wolf
Quantum tomography under prior information, *Communications in Mathematical Physics* 318(2), 355–374, 2013.

Aldo Conca, Dan Edidin, Milena Hering and Cynthia Vinzant
An algebraic characterization of injectivity in phase retrieval, *Applied and Computational Harmonic Analysis* 38(2), 346–356, 2015.

Franz J. Király and Martin Ehler
The algebraic approach to phase retrieval and explicit inversion at the identifiability threshold, *arXiv:1402.4053*, 2014.

Stability of phase retrieval

Definition

For a frame Φ, the associated measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ is called stable with a constant C in a set $T \subset \mathbb{C}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta \in [0,2\pi)} ||x - e^{i\theta} y||_2 ||x + e^{i\theta} y||_2.$$

Stability of phase retrieval

Definition

For a frame Φ, the associated measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ is called stable with a constant C in a set $T \subset \mathbb{C}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta \in [0,2\pi)} ||x - e^{i\theta} y||_2 ||x + e^{i\theta} y||_2.$$

Note: *Stability in a set is a stronger property than injectivity.*

Stability of phase retrieval

Definition

For a frame Φ, the associated measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ is called stable with a constant C in a set $T \subset \mathbb{C}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta \in [0,2\pi)} ||x - e^{i\theta} y||_2 ||x + e^{i\theta} y||_2.$$

Note: *Stability in a set is a stronger property than injectivity.*

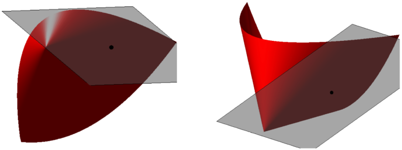


Figure: Let $\mathbf{A}_\Phi(X) = \{\mathrm{Tr}(X \varphi_j \varphi_j^*)\}_{j=1}^N$, so that $\mathcal{A}_\Phi(x) = \mathbf{A}_\Phi(xx^*)$.

Figure courtesy: E. J. Candès, T. Strohmer and V. Voroninski, 2013.

Stability of phase retrieval

Definition

For a frame Φ, the associated measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ is called stable with a constant C in a set $T \subset \mathbb{C}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta \in [0,2\pi)} ||x - e^{i\theta} y||_2 ||x + e^{i\theta} y||_2.$$

Note: *Stability in a set is a stronger property than injectivity.*

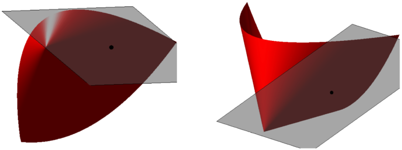


Figure: Let $\mathbf{A}_\Phi(X) = \{\mathrm{Tr}(X\varphi_j\varphi_j^*)\}_{j=1}^N$, so that $\mathcal{A}_\Phi(x) = \mathbf{A}_\Phi(xx^*)$. The positive semidefinite cone $\{X \succeq 0\}$ and the affine space $\{\mathbf{A}_\Phi(X) = b\}$ in $\mathbb{R}^3$ are tangent to each other at the rank 1 matrix $X = xx^*$. Figure courtesy: E. J. Candès, T. Strohmer and V. Voroninski, 2013.

Stability of phase retrieval

- (Eldar and Mendelson) For a frame Φ of cardinality $O(M)$, such that $\varphi_j(m)$ are independent L-subgaussian, the mapping $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ under the additional small ball assumption on the distribution of $\varphi_j(m)$.

Yonina C. Eldar and Shahar Mendelson
Phase retrieval: Stability and recovery guarantees, *Applied and Computational Harmonic Analysis* 36(3), 473–494, 2014.

Stability of phase retrieval

- (Eldar and Mendelson) For a frame Φ of cardinality $O(M)$, such that $\varphi_j(m)$ are independent L-subgaussian, the mapping $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ under the additional small ball assumption on the distribution of $\varphi_j(m)$.
- (Krahmer and Liu) The small ball assumption can be dropped if we restrict to stability in the set of μ-flat vectors $T_\mu = \{x \in \mathbb{R}^M, ||x||_\infty \leq \mu ||x||_2\}$.
- (Kabanava, Kueng, Rauhut, and Terstiege) Map $\mathcal{A}_\Phi$ with frame vectors independently uniformly sampled from an approximate 4-design is stable in $\mathbb{C}^M$.
- (Kueng, Zhu, and Gross) Map $\mathcal{A}_\Phi$ with frame vectors independently uniformly sampled from Clifford orbit is stable in $\mathbb{C}^M$.

Yonina C. Eldar and Shahar Mendelson
Phase retrieval: Stability and recovery guarantees, *Applied and Computational Harmonic Analysis* 36(3), 473–494, 2014.

Felix Krahmer and Yi-Kai Liu
Phase retrieval without small-ball probability assumptions, *IEEE Transactions on Information Theory* 64(1), 485–500, 2017.

Maryia Kabanava, Richard Kueng, Holger Rauhut and Ulrich Terstiege
Stable low-rank matrix recovery via null space properties, *Information and Inference: A Journal of the IMA* 5(4), 405–441, 2016.

Richard Kueng, Huangjun Zhu and David Gross
Low rank matrix recovery from Clifford orbits, *arXiv:1610.08070*, 2016.

Stability of phase retrieval

Common condition in the mentioned results: independent frame vectors.

Stability of phase retrieval

Common condition in the mentioned results: independent frame vectors.

Question

What properties of the measurement frame imply stability of the associated phaseless measurement map?

Stability of phase retrieval

Common condition in the mentioned results: independent frame vectors.

Question

What properties of the measurement frame imply stability of the associated phaseless measurement map?

Goal: Propose a new method that can be used to establish stability of the measurement maps for larger classes of frames, including frames with correlated frame vectors. In particular, for structures frames arising in phase retrieval applications, such as Gabor frames.

Stability of phase retrieval

Common condition in the mentioned results: independent frame vectors.

Question

What properties of the measurement frame imply stability of the associated phaseless measurement map?

Goal: Propose a new method that can be used to establish stability of the measurement maps for larger classes of frames, including frames with correlated frame vectors. In particular, for structures frames arising in phase retrieval applications, such as Gabor frames.

Approach: Frame order statistics: if frame vectors are well spread in $\mathbb{C}^M$, then for each one-dimensional subspace in $\mathbb{C}^M$, there are not too many frame vectors that are almost colinear or almost orthogonal to it.

Frame order statistics: definition

Definition (Frame order statistics)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{S}^{M-1}$.

Frame order statistics: definition

Definition (Frame order statistics)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{S}^{M-1}$.

1. For $\alpha \leq N$, the α-smallest frame order statistics of Φ is given by

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) = \max_{\substack{J \subseteq \{1,\dots,N\}, \\ |J| \geq \alpha}} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

Frame order statistics: definition

Definition (Frame order statistics)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{S}^{M-1}$.

1. For $\alpha \leq N$, the α-smallest frame order statistics of Φ is given by

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) = \max_{\substack{J \subseteq \{1,\dots,N\},\\ |J| \geq \alpha}} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

2. For $\beta \leq N$, the β-largest frame order statistics of Φ is given by

$$\mathcal{L}_{FOS}(\Phi, \beta, x) = \min_{\substack{J \subseteq \{1,\dots,N\},\\ |J| \geq \beta}} \max_{j \in J} |\langle x, \varphi_j \rangle|.$$

Frame order statistics: definition

Definition (Frame order statistics)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{S}^{M-1}$.

1. For $\alpha \leq N$, the α-smallest frame order statistics of Φ is given by

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) = \max_{\substack{J \subseteq \{1,\dots,N\}, \\ |J| \geq \alpha}} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

2. For $\beta \leq N$, the β-largest frame order statistics of Φ is given by

$$\mathcal{L}_{FOS}(\Phi, \beta, x) = \min_{\substack{J \subseteq \{1,\dots,N\}, \\ |J| \geq \beta}} \max_{j \in J} |\langle x, \varphi_j \rangle|.$$

If we delete $\lfloor N - \alpha \rfloor$ smallest and $\lfloor N - \beta \rfloor$ largest in modulus frame coefficients, then the remaining ones satisfy

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) \leq |\langle x, \varphi_j \rangle| \leq \mathcal{L}_{FOS}(\Phi, \beta, x).$$

Frame order statistics: results

Theorem (Pfander, S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame. Then the following holds:

1. *Suppose $\varphi_j(m)$ are i.i.d. centered random variables with bounded fourth moment and $N = O(M \log M)$. Then for each $\alpha, \beta \in (0,1)$, there exist constants $c \in (0,1)$, $K > 1$, such that with high probability*

$$\frac{c}{\sqrt{M}} \leq \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha|\Phi|, x) \leq \max_{x \in \mathbb{S}^{M-1}} \mathcal{L}_{FOS}(\Phi, \beta|\Phi|, x) \leq \frac{K}{\sqrt{M}}.$$

Götz E Pfander and Palina Salanevich
Robust phase retrieval algorithm for time-frequency structured measurements, *SIAM Journal on Imaging Sciences* 12(2), 736–761, 2019.

Frame order statistics: results

Theorem (Pfander, S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ *be a frame. Then the following holds:*

1. *Suppose* $\varphi_j(m)$ *are i.i.d. centered random variables with bounded fourth moment and* $N = O(M \log M)$*. Then for each* $\alpha, \beta \in (0,1)$*, there exist constants* $c \in (0,1)$*,* $K > 1$*, such that with high probability*

$$\frac{c}{\sqrt{M}} \leq \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha|\Phi|, x) \leq \max_{x \in \mathbb{S}^{M-1}} \mathcal{L}_{FOS}(\Phi, \beta|\Phi|, x) \leq \frac{K}{\sqrt{M}}.$$

2. *Fix* $x \in \mathbb{S}^{M-1}$ *and suppose* $\varphi_j \sim$ *Unif.*$(\mathbb{S}^{M-1})$*. Then for any* $\epsilon, \eta \in (0,1)$*,* $\epsilon + \eta < 1$*, there exist* $c \in (0,1)$ *and* $K > 1$*, such that, with high probability,*

$$\frac{c}{\sqrt{M}} \leq \mathcal{S}_{FOS}\left(\Phi, \epsilon|\Phi|, x\right) \leq \mathcal{L}_{FOS}\left(\Phi, \eta|\Phi|, x\right) \leq \frac{K}{\sqrt{M}}.$$

Götz E Pfander and Palina Salanevich
Robust phase retrieval algorithm for time-frequency structured measurements, *SIAM Journal on Imaging Sciences* 12(2), 736–761, 2019.

Stability using Frame order statistics bounds

Bounds on Frame order statistics
Essentially flat frame coefficients vector

→

Stable phaseles map $\mathcal{A}_\Phi$
Unique solution to phase retrieval problem

Stability using Frame order statistics bounds

Bounds on Frame order statistics
Essentially flat frame coefficients vector

→

Stable phaseles map $\mathcal{A}_\Phi$
Unique solution to phase retrieval problem

Theorem (S.)

Let $\Phi \subset \mathbb{C}^M$ be a frame. Fix $\alpha < 1 - \frac{1}{2C_0}$ and denote

$$S_\alpha(M) = \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha|\Phi|, x).$$

Then the phaseless measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq (2\alpha - 1)|\Phi| S_\alpha(M)^2$ in $\mathbb{C}^M$.

Stability using Frame order statistics bounds

Bounds on Frame order statistics
Essentially flat frame coefficients vector

→

Stable phaseles map $\mathcal{A}_\Phi$
Unique solution to phase retrieval problem

Theorem (S.)

Let $\Phi \subset \mathbb{C}^M$ be a frame. Fix $\alpha < 1 - \frac{1}{2C_0}$ and denote

$$S_\alpha(M) = \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha|\Phi|, x).$$

Then the phaseless measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq (2\alpha - 1)|\Phi| S_\alpha(M)^2$ in $\mathbb{C}^M$.

Note: We need $S_\alpha(M) \geq \frac{c}{\sqrt{|\Phi|}}$, for some $c > 0$ to insure that C is bounded away from zero for all M.

Corollary: stability with independent frame vectors

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^{N} \subset \mathbb{C}^M$ be a frame. Suppose $\varphi_j(m)$ are i.i.d. centered random variables with bounded fourth moment and $N = O(M \log M)$. Then there exists a numerical constant $L > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq L \log(M)$.

Corollary: stability with independent frame vectors

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame. Suppose $\varphi_j(m)$ are i.i.d. centered random variables with bounded fourth moment and $N = O(M \log M)$. Then there exists a numerical constant $L > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq L \log(M)$.

Compare to previous results:

1. Stability for a larger class of random frames Φ (more general distribution).
2. No additional restrictions on the set T of the measured signals.

Corollary: stability with independent frame vectors

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame. Suppose $\varphi_j(m)$ are i.i.d. centered random variables with bounded fourth moment and $N = O(M \log M)$. Then there exists a numerical constant $L > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq L \log(M)$.

Compare to previous results:

1. Stability for a larger class of random frames Φ (more general distribution).
2. No additional restrictions on the set T of the measured signals.
3. **Cost:** larger frame cardinality ($O(M \log(M))$ instead of $O(M)$).

Corollary: stability without independence assumption

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ *be a frame such that* $\varphi_j \sim Unif.(\mathbb{S}^{M-1})$*. Then there exists a numerical constant* C*, such that for each pair* $x, y \in \mathbb{C}^M$ *the following holds with high probability*

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C\frac{|\Phi|}{M} \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

Corollary: stability without independence assumption

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame such that $\varphi_j \sim Unif.(\mathbb{S}^{M-1})$. Then there exists a numerical constant C, such that for each pair $x, y \in \mathbb{C}^M$ the following holds with high probability

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C\frac{|\Phi|}{M} \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

Note: a non-unifrom stability result. For any pair of signals $x, y \in \mathbb{C}^M$, the phaseless measurement map $\mathcal{A}_\Phi$ distinguishes between them with high probability.

Corollary: stability without independence assumption

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame such that $\varphi_j \sim \text{Unif.}(\mathbb{S}^{M-1})$. Then there exists a numerical constant C, such that for each pair $x, y \in \mathbb{C}^M$ the following holds with high probability

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C\frac{|\Phi|}{M} \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

Note: a non-unifrom stability result. For any pair of signals $x, y \in \mathbb{C}^M$, the phaseless measurement map $\mathcal{A}_\Phi$ distinguishes between them with high probability.

In particular: This result holds for a Gabor frame $\Phi = (g, \Lambda)$ with window $g \sim \text{Unif.}(\mathbb{S}^{M-1})$, and $|\Lambda| > M$.

Corollary: stability without independence assumption

Corollary (S.)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame such that $\varphi_j \sim \text{Unif.}(\mathbb{S}^{M-1})$. Then there exists a numerical constant C, such that for each pair $x, y \in \mathbb{C}^M$ the following holds with high probability

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C\frac{|\Phi|}{M} \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

Note: a non-unifrom stability result. For any pair of signals $x, y \in \mathbb{C}^M$, the phaseless measurement map $\mathcal{A}_\Phi$ distinguishes between them with high probability.

In particular: This result holds for a Gabor frame $\Phi = (g, \Lambda)$ with window $g \sim \text{Unif.}(\mathbb{S}^{M-1})$, and $|\Lambda| > M$.

Question

Can we further exploit the structure of Gabor frames to improve this result and show uniform stability?

Phase retrieval with Gabor frames

Theorem (S.)

Let $\Phi = (g, \Lambda)$ be a full Gabor frame with window $g \sim Unif.(\mathbb{S}^{M-1})$ and $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$. Then there exists a numerical constant $C > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant C.

Phase retrieval with Gabor frames

Theorem (S.)

Let $\Phi = (g, \Lambda)$ be a full Gabor frame with window $g \sim Unif.(\mathbb{S}^{M-1})$ and $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$. Then there exists a numerical constant $C > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant C.

1. Reducing the cardinality of Λ is a complicated task.

Phase retrieval with Gabor frames

Theorem (S.)

Let $\Phi = (g, \Lambda)$ be a full Gabor frame with window $g \sim Unif.(\mathbb{S}^{M-1})$ and $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$. Then there exists a numerical constant $C > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant C.

1. Reducing the cardinality of Λ is a complicated task.
2. Only the injectivity of Gabor frames, has been addressed before, and exclusively in the case when $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$.

Irena Bojarovska and Axel Flinth
Phase retrieval from Gabor measurements, *Journal of Fourier Analysis and Applications* 22(3), 542 – 567, 2016.

Phase retrieval with Gabor frames

Theorem (S.)

Let $\Phi = (g, \Lambda)$ be a full Gabor frame with window $g \sim Unif.(\mathbb{S}^{M-1})$ and $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$. Then there exists a numerical constant $C > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant C.

1. Reducing the cardinality of Λ is a complicated task.
2. Only the injectivity of Gabor frames, has been addressed before, and exclusively in the case when $\Lambda = \mathbb{Z}_M \times \mathbb{Z}_M$.
3. Relies of the particular structure of a Gabor frame, and on the uniform bound on its $\mathcal{L}_{FOS}$ (Pfander, S.):

$$\mathbb{P}\left(\max_{x \in \mathbb{S}^{M-1}} \mathcal{L}_{FOS}\left((g, \Lambda), \frac{cM}{\log^4 M}, x\right) < \sqrt{\frac{3}{2c}\frac{\log^2 M}{\sqrt{M}}}\right) \geq 1 - e^{-c_1 \log^3 M}.$$

Irena Bojarovska and Axel Flinth
Phase retrieval from Gabor measurements, *Journal of Fourier Analysis and Applications* 22(3), 542 – 567, 2016.

Phase retrieval algorithm for structured frames

Question

Given a frame Φ with associated phaseless measurement map $\mathcal{A}_\Phi$ being injective/stable, how to efficiently reconstruct x from $\mathcal{A}_\Phi(x)$?

Phase retrieval algorithm for structured frames

Question

Given a frame Φ with associated phaseless measurement map $\mathcal{A}_\Phi$ being injective/stable, how to efficiently reconstruct x from $\mathcal{A}_\Phi(x)$?

- Mostly studied for Gaussian frames: PhaseLift, Wirtinger flow, ...
- Very little is known for structured, application relevant frames.

Phase retrieval algorithm for structured frames

Question

Given a frame Φ with associated phaseless measurement map $\mathcal{A}_\Phi$ being injective/stable, how to efficiently reconstruct x from $\mathcal{A}_\Phi(x)$?

- Mostly studied for Gaussian frames: PhaseLift, Wirtinger flow, ...
- Very little is known for structured, application relevant frames.

  Idea: bounds on frame order statistics can help in showing reconstruction and robustness guarantees.

Phase retrieval algorithm for structured frames

Question

Given a frame Φ with associated phaseless measurement map $\mathcal{A}_\Phi$ being injective/stable, how to efficiently reconstruct x from $\mathcal{A}_\Phi(x)$?

- Mostly studied for Gaussian frames: PhaseLift, Wirtinger flow, ...
- Very little is known for structured, application relevant frames.

  Idea: bounds on frame order statistics can help in showing reconstruction and robustness guarantees.

Phaseless masked Fourier measurement procedure:

$$\mathcal{A}(x) = \{|\mathcal{F}(x \odot g_k)(\ell)|^2\}_{k \in K, \ell \in \mathbb{Z}_M} = \{|\langle x, M_\ell \overline{g_k} \rangle|^2\}_{k \in K, \ell \in \mathbb{Z}_M}$$

Phase retrieval algorithm for structured frames

Question

Given a frame Φ with associated phaseless measurement map $\mathcal{A}_\Phi$ being injective/stable, how to efficiently reconstruct x from $\mathcal{A}_\Phi(x)$?

- Mostly studied for Gaussian frames: PhaseLift, Wirtinger flow, ...
- Very little is known for structured, application relevant frames.

  Idea: bounds on frame order statistics can help in showing reconstruction and robustness guarantees.

Phaseless masked Fourier measurement procedure:

$$\mathcal{A}(x) = \{|\mathcal{F}(x \odot g_k)(\ell)|^2\}_{k \in K, \ell \in \mathbb{Z}_M} = \{|\langle x, M_\ell \overline{g_k} \rangle|^2\}_{k \in K, \ell \in \mathbb{Z}_M}$$

Let $\{g_k\}_{k \in K} \subset \mathbb{S}^{M-1}$, with $K \subset \mathbb{Z}_M$ and $|K|$ being a constant independent of M, be a set of masks that is constructed in one of the following two ways:

1. $g_k \sim$ i.i.d. Unif.$(\mathbb{S}^{M-1})$;
2. $g_k = T_k g$, where $g \sim$ Unif.$(\mathbb{S}^{M-1})$.

Phase retrieval algorithm for structured frames

Theorem (Pfander, S.)

Fix $x \in \mathbb{C}^M$, and let $\{g_k\}_{k \in K}$ be as above. Then there exist a set of additional masks $\{g_t\}_{t \in T}$ with $|T| = O(\log(M))$, and a reconstruction algorithm, such that the estimate $\tilde{x}$ produced by it from the measurements with masks $\{g_t\}_{t \in K \cup T}$ satisfies

$$\min_{\theta \in [0, 2\pi)} ||\tilde{x} - e^{i\theta} x||_2^2 \leq C\sqrt{M} ||\nu||_2,$$

with overwhelming probability, provided the noise vector ν satisfies $\frac{||\nu||_2}{||x||_2^2} \leq \frac{c}{M}$.

Phase retrieval algorithm for structured frames

Theorem (Pfander, S.)

Fix $x \in \mathbb{C}^M$, and let $\{g_k\}_{k\in K}$ be as above. Then there exist a set of additional masks $\{g_t\}_{t\in T}$ with $|T| = O(\log(M))$, and a reconstruction algorithm, such that the estimate $\tilde{x}$ produced by it from the measurements with masks $\{g_t\}_{t\in K\cup T}$ satisfies

$$\min_{\theta\in[0,2\pi)} ||\tilde{x} - e^{i\theta}x||_2^2 \leq C\sqrt{M}||\nu||_2,$$

with overwhelming probability, provided the noise vector ν satisfies $\frac{||\nu||_2}{||x||_2^2} \leq \frac{c}{M}$.

1. Not just an existence result: the algorithm and the set of additional masks $\{g_t\}_{t\in T}$ construction are provided.

Phase retrieval algorithm for structured frames

Theorem (Pfander, S.)

Fix $x \in \mathbb{C}^M$, and let $\{g_k\}_{k\in K}$ be as above. Then there exist a set of additional masks $\{g_t\}_{t\in T}$ with $|T| = O(\log(M))$, and a reconstruction algorithm, such that the estimate $\tilde{x}$ produced by it from the measurements with masks $\{g_t\}_{t\in K\cup T}$ satisfies

$$\min_{\theta\in[0,2\pi)} ||\tilde{x} - e^{i\theta}x||_2^2 \leq C\sqrt{M}||\nu||_2,$$

with overwhelming probability, provided the noise vector ν satisfies $\frac{||\nu||_2}{||x||_2^2} \leq \frac{c}{M}$.

1. Not just an existence result: the algorithm and the set of additional masks $\{g_t\}_{t\in T}$ construction are provided.
2. In the case when $g_k = T_k g$, where $g \sim \text{Unif.}(\mathbb{S}^{M-1})$, the set of additional masks is also formed as time shifts of a modified window. In this case, the measurement frame is a union of two Gabor frames.

Idea of the polarization approach

Let $\Phi_\Lambda = (g, \Lambda)$ with g *uniformly distributed* on the unit sphere $\mathbb{S}^{M-1} \subset \mathbb{C}^M$ and $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$ with $|F|$ being a constant not depending on M.

Idea of the polarization approach

Let $\Phi_\Lambda = (g, \Lambda)$ with g *uniformly distributed* on the unit sphere $\mathbb{S}^{M-1} \subset \mathbb{C}^M$ and $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$ with $|F|$ being a constant not depending on M.

Have: phaseless measurements $b_\lambda = |\langle x, \pi(\lambda) g\rangle|^2$, $\lambda \in \Lambda$.

Idea of the polarization approach

Let $\Phi_\Lambda = (g, \Lambda)$ with g *uniformly distributed* on the unit sphere $\mathbb{S}^{M-1} \subset \mathbb{C}^M$ and $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$ with $|F|$ being a constant not depending on M.

Have: phaseless measurements $b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2$, $\lambda \in \Lambda$.

Aim to reconstruct: phases of frame coefficients $u_\lambda = \frac{\langle x,\pi(\lambda)g\rangle}{|\langle x,\pi(\lambda)g\rangle|}$, $\lambda \in \Lambda$ with $b_\lambda \neq 0$.

Idea of the polarization approach

Let $\Phi_\Lambda = (g, \Lambda)$ with g *uniformly distributed* on the unit sphere $\mathbb{S}^{M-1} \subset \mathbb{C}^M$ and $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$ with $|F|$ being a constant not depending on M.

Have: phaseless measurements $b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2$, $\lambda \in \Lambda$.

Aim to reconstruct: phases of frame coefficients $u_\lambda = \frac{\langle x, \pi(\lambda)g\rangle}{|\langle x, \pi(\lambda)g\rangle|}$, $\lambda \in \Lambda$ with $b_\lambda \neq 0$.

Suppose in addition to phaseless measurements b we know *relative phases* between frame coefficients

$$\omega_{\lambda_1\lambda_2} = u_{\lambda_1}^{-1} u_{\lambda_2} = \frac{\overline{\langle x, \pi(\lambda_1)g\rangle}\langle x, \pi(\lambda_2)g\rangle}{|\langle x, \pi(\lambda_1)g\rangle||\langle x, \pi(\lambda_2)g\rangle|}, \ (\lambda_1, \lambda_2) \in E,$$

defined for $b_{\lambda_1}, b_{\lambda_2} \neq 0$. Here $E \subset \Lambda \times \Lambda$ to be chosen later.

Polarization identity

Lemma (Polarization identity)

Let $\omega = e^{2\pi i/3}$. *For any* $\lambda_1, \lambda_2 \in \Lambda$, *such that* $b_{\lambda_1}, b_{\lambda_2} \neq 0$,

$$\omega_{\lambda_1\lambda_2} = \frac{1}{3|\langle x, \pi(\lambda_1)g\rangle||\langle x, \pi(\lambda_2)g\rangle|} \sum_{t=0}^{2} \omega^t \left|\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\rangle\right|^2.$$

Polarization identity

Lemma (Polarization identity)

Let $\omega = e^{2\pi i/3}$. For any $\lambda_1, \lambda_2 \in \Lambda$, such that $b_{\lambda_1}, b_{\lambda_2} \neq 0$,

$$\omega_{\lambda_1 \lambda_2} = \frac{1}{3|\langle x, \pi(\lambda_1)g\rangle||\langle x, \pi(\lambda_2)g\rangle|} \sum_{t=0}^{2} \omega^t \big|\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\rangle\big|^2.$$

We take measurements with respect to the union of two frames:

$$\Phi_\Lambda \cup \Phi_E = (g, \Lambda) \cup \{\pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\}_{t \in \{0,1,2\},\ (\lambda_1,\lambda_2) \in E}.$$

Polarization identity

Lemma (Polarization identity)

Let $\omega = e^{2\pi i/3}$*. For any* $\lambda_1, \lambda_2 \in \Lambda$*, such that* $b_{\lambda_1}, b_{\lambda_2} \neq 0$*,*

$$\omega_{\lambda_1\lambda_2} = \frac{1}{3|\langle x, \pi(\lambda_1)g\rangle||\langle x, \pi(\lambda_2)g\rangle|} \sum_{t=0}^{2} \omega^t \left|\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\rangle\right|^2.$$

We take measurements with respect to the union of two frames:

$$\Phi_\Lambda \cup \Phi_E = (g, \Lambda) \cup \{\pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\}_{t\in\{0,1,2\},\ (\lambda_1,\lambda_2)\in E}.$$

The additional measurements are masked Fourier transform coefficients:

$$b_{\lambda_1\lambda_2 t} = |\langle x, \pi(k_1, \ell_1)g + \omega^t \pi(k_2, \ell_2)g\rangle|^2 = \left|\mathcal{F}\left(x \odot \bar{p}_{\ell_2-\ell_1,k_1,k_2}(t) \odot T_{k_1}\bar{g}\right)(\ell_1)\right|^2,$$

where $p_{c,k_1,k_2}(t)(m) = 1 + e^{2\pi i\left(\frac{cm}{M}+\frac{t}{3}\right)} \frac{g(m-k_2)}{g(m-k_1)}$, $m \in \mathbb{Z}_M$.

Polarization approach: phase propagation algorithm

Algorithm 1: Phase propagation algorithm

Input : for given $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, $E \subset \Lambda \times \Lambda$, and $g \in \mathbb{C}^M$, measurements of the form $\{b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2\}_{\lambda\in\Lambda}$, $\left\{\omega_{\lambda_1\lambda_2} = \frac{\overline{\langle x,\pi(\lambda_1)g\rangle}\langle x,\pi(\lambda_2)g\rangle}{|\langle x,\pi(\lambda_1)g\rangle||\langle x,\pi(\lambda_2)g\rangle|}\right\}_{(\lambda_1,\lambda_2)\in E}$.

Output: $\tilde{x} = (\Phi_\Lambda\Phi_\Lambda^*)^{-1}\Phi_\Lambda c \in [x]$ for $c = \{c_\lambda\}_{\lambda\in\Lambda}$.

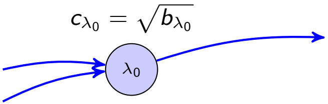
Polarization approach: phase propagation algorithm

Algorithm 1: Phase propagation algorithm

Input : for given $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, $E \subset \Lambda \times \Lambda$, and $g \in \mathbb{C}^M$, measurements of the form $\{b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2\}_{\lambda\in\Lambda}$, $\left\{\omega_{\lambda_1\lambda_2} = \frac{\overline{\langle x,\pi(\lambda_1)g\rangle}\langle x,\pi(\lambda_2)g\rangle}{|\langle x,\pi(\lambda_1)g\rangle||\langle x,\pi(\lambda_2)g\rangle|}\right\}_{(\lambda_1,\lambda_2)\in E}$.

Output: $\tilde{x} = (\Phi_\Lambda\Phi_\Lambda^*)^{-1}\Phi_\Lambda c \in [x]$ for $c = \{c_\lambda\}_{\lambda\in\Lambda}$.

1: choose λ_0 with $b_{\lambda_0} \neq 0$, set $c_{\lambda_0} = \sqrt{b_{\lambda_0}}$; for all $\lambda \in \Lambda$, s.t. $b_\lambda = 0$, set $c_\lambda = 0$;



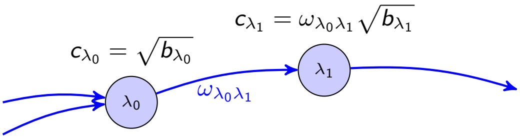
Polarization approach: phase propagation algorithm

Algorithm 1: Phase propagation algorithm

Input : for given $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, $E \subset \Lambda \times \Lambda$, and $g \in \mathbb{C}^M$, measurements of the form $\{b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2\}_{\lambda\in\Lambda}$, $\left\{\omega_{\lambda_1\lambda_2} = \frac{\overline{\langle x,\pi(\lambda_1)g\rangle}\langle x,\pi(\lambda_2)g\rangle}{|\langle x,\pi(\lambda_1)g\rangle||\langle x,\pi(\lambda_2)g\rangle|}\right\}_{(\lambda_1,\lambda_2)\in E}$.

Output: $\tilde{x} = (\Phi_\Lambda\Phi_\Lambda^*)^{-1}\Phi_\Lambda c \in [x]$ for $c = \{c_\lambda\}_{\lambda\in\Lambda}$.

choose λ_0 with $b_{\lambda_0} \neq 0$, set $c_{\lambda_0} = \sqrt{b_{\lambda_0}}$; for all $\lambda \in \Lambda$, s.t. $b_\lambda = 0$, set $c_\lambda = 0$;
while not all c_λ are set **do**
    choose $c_{\lambda_1} \neq 0$ already known;
end while



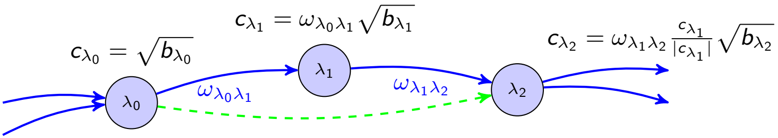
Polarization approach: phase propagation algorithm

Algorithm 1: Phase propagation algorithm

Input : for given $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, $E \subset \Lambda \times \Lambda$, and $g \in \mathbb{C}^M$, measurements of the form $\{b_\lambda = |\langle x, \pi(\lambda)g\rangle|^2\}_{\lambda\in\Lambda}$, $\left\{\omega_{\lambda_1\lambda_2} = \frac{\overline{\langle x,\pi(\lambda_1)g\rangle}\langle x,\pi(\lambda_2)g\rangle}{|\langle x,\pi(\lambda_1)g\rangle||\langle x,\pi(\lambda_2)g\rangle|}\right\}_{(\lambda_1,\lambda_2)\in E}$.

Output: $\tilde{x} = (\Phi_\Lambda\Phi_\Lambda^*)^{-1}\Phi_\Lambda c \in [x]$ for $c = \{c_\lambda\}_{\lambda\in\Lambda}$.

```
choose λ0 with b_{λ0} ≠ 0, set c_{λ0} = sqrt(b_{λ0}); for all λ ∈ Λ, s.t. b_λ = 0, set c_λ = 0;
while not all c_λ are set do
    choose c_{λ1} ≠ 0 already known;
    for λ2, s.t. (λ1, λ2) ∈ E and c_{λ2} is not set, do
        set c_{λ2} = ω_{λ1λ2} (c_{λ1}/|c_{λ1}|) sqrt(b_{λ2}).
    end for
end while
```



Reconstruction in the case of noisy measurements

Algorithm 2: Phaseless reconstruction in the noisy case

Input : phaseless measurements $b = \mathcal{A}_{\Phi_\Lambda \cup \Phi_E}(x)$;
parameters $\tau_0, \alpha, \beta \in (0,1)$.

Output: $\tilde{x} \in [x]$, initial signal up to a global phase.

1: construct the graph of measurements $G = (\Lambda, E)$;
2: assign to each vertex $\lambda \in \Lambda$ the weight b_λ and to each edge $(\lambda_1, \lambda_2) \in E$ the weight $\omega_{\lambda_1 \lambda_2}$;
3: delete $(1-\alpha)|\Lambda|$ vertices with the smallest weights and $(1-\beta)|\Lambda|$ vertices with the largest weights to obtain $G' = (\Lambda', E') \subset G$;
4: choose $G'' = (\Lambda'', E'') \subset G'$ with $\mathrm{spg}(G'') > \tau_0$ (spectral clustering);
5: use angular synchronization procedure to obtain $c_\lambda = \tilde{u}_\lambda \sqrt{b_\lambda}$, $\lambda \in \Lambda''$;
6: reconstruct $\tilde{x} = (\Phi_{\Lambda''} \Phi^*_{\Lambda''})^{-1} \Phi_{\Lambda''} c$ from $c = \{c_\lambda\}_{\lambda \in \Lambda''}$.

Conclusions

1. Frame order statistics provide a general tool that can be used to show stability of the phaseless measurement maps associated to a wide range of different classes of frames, including frame with correlated frame vectors.

Conclusions

1. Frame order statistics provide a general tool that can be used to show stability of the phaseless measurement maps associated to a wide range of different classes of frames, including frame with correlated frame vectors.
2. Obtaining uniform bounds on the frame order statistics of Gabor frames would imply stability, and thus also injectivity, of $\mathcal{A}_{(g,\Lambda)}$ with a random Gabor window g, for any $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ with $|\Lambda|$ sufficiently large. This would be a big step forward in study of phase retrieval with Gabor frames.

Conclusions

1. Frame order statistics provide a general tool that can be used to show stability of the phaseless measurement maps associated to a wide range of different classes of frames, including frame with correlated frame vectors.
2. Obtaining uniform bounds on the frame order statistics of Gabor frames would imply stability, and thus also injectivity, of $\mathcal{A}_{(g,\Lambda)}$ with a random Gabor window g, for any $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$ with $|\Lambda|$ sufficiently large. This would be a big step forward in study of phase retrieval with Gabor frames.
3. Study of the frame order statistics is also important for other problems in image processing, including quantization.

Thank You for Your Attention!

Proof

Let $|\Phi| = N$. For any $x, y \in \mathbb{C}^M$, let $\theta_{xy} \in [0, 2\pi)$ be such that

$||\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|| = |\langle x - e^{i\theta_{xy}} y, \varphi_i\rangle|$ and $||\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|| = |\langle x + e^{i\theta_{xy}} y, \varphi_i\rangle|$.

Proof

Let $|\Phi| = N$. For any $x, y \in \mathbb{C}^M$, let $\theta_{xy} \in [0, 2\pi)$ be such that

$||\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|| = |\langle x - e^{i\theta_{xy}} y, \varphi_i\rangle|$ and $||\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|| = |\langle x + e^{i\theta_{xy}} y, \varphi_i\rangle|$.

Then $||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 = \sum_{i=1}^N |\left(|\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|\right)\left(|\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|\right)| =$
$||x - e^{i\theta_{xy}} y||_2 ||x + e^{i\theta_{xy}} y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right|$.

Proof

Let $|\Phi| = N$. For any $x, y \in \mathbb{C}^M$, let $\theta_{xy} \in [0, 2\pi)$ be such that $||\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|| = |\langle x - e^{i\theta_{xy}} y, \varphi_i\rangle|$ and $||\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|| = |\langle x + e^{i\theta_{xy}} y, \varphi_i\rangle|$.

Then $||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 = \sum_{i=1}^N |\left(|\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|\right)\left(|\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|\right)| =$ $||x - e^{i\theta_{xy}} y||_2 ||x + e^{i\theta_{xy}} y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right|$.

Let us fix some $\frac{1}{2} < \alpha < 1 - \frac{1}{2C_0}$. Then, for unit vectors $u = \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}$ and $v = \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}$, there exist $J_u, J_v \subset \{1, \ldots, N\}$ with $|J_u|, |J_v| \geq \alpha N$ and $|\langle u, \varphi_i\rangle| |\langle v, \varphi_i\rangle| \geq \frac{c^2}{M}$ for all $j \in J_u \cap J_v$.

Proof

Let $|\Phi| = N$. For any $x, y \in \mathbb{C}^M$, let $\theta_{xy} \in [0, 2\pi)$ be such that

$||\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|| = |\langle x - e^{i\theta_{xy}} y, \varphi_i\rangle|$ and $||\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|| = |\langle x + e^{i\theta_{xy}} y, \varphi_i\rangle|$.

Then $||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 = \sum_{i=1}^N |\left(|\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|\right)\left(|\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|\right)| =$
$||x - e^{i\theta_{xy}} y||_2 ||x + e^{i\theta_{xy}} y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right|$.

Let us fix some $\frac{1}{2} < \alpha < 1 - \frac{1}{2C_0}$. Then, for unit vectors $u = \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}$ and $v = \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}$, there exist $J_u, J_v \subset \{1, \dots, N\}$ with $|J_u|, |J_v| \geq \alpha N$ and $|\langle u, \varphi_i\rangle| \, |\langle v, \varphi_i\rangle| \geq \frac{c^2}{M}$ for all $j \in J_u \cap J_v$. Then, since $|J_u \cap J_v| \geq (2\alpha - 1)N > 0$,

$$\sum_{i=1}^N |\langle u, \varphi_i\rangle||\langle v, \varphi_i\rangle| \geq \sum_{i \in J_u \cap J_v} |\langle u, \varphi_i\rangle||\langle v, \varphi_i\rangle| \geq \frac{c^2(2\alpha - 1)N}{M}.$$

Proof

Let $|\Phi| = N$. For any $x, y \in \mathbb{C}^M$, let $\theta_{xy} \in [0, 2\pi)$ be such that
$||\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|| = |\langle x - e^{i\theta_{xy}} y, \varphi_i\rangle|$ and $||\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|| = |\langle x + e^{i\theta_{xy}} y, \varphi_i\rangle|$.

Then $||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 = \sum_{i=1}^N |\left(|\langle x, \varphi_i\rangle| - |\langle y, \varphi_i\rangle|\right)\left(|\langle x, \varphi_i\rangle| + |\langle y, \varphi_i\rangle|\right)| =$
$||x - e^{i\theta_{xy}} y||_2 ||x + e^{i\theta_{xy}} y||_2 \sum_{i=1}^N \left|\left\langle \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}, \varphi_i \right\rangle\right|$.

Let us fix some $\frac{1}{2} < \alpha < 1 - \frac{1}{2C_0}$. Then, for unit vectors $u = \frac{x - e^{i\theta_{xy}} y}{||x - e^{i\theta_{xy}} y||_2}$ and $v = \frac{x + e^{i\theta_{xy}} y}{||x + e^{i\theta_{xy}} y||_2}$, there exist $J_u, J_v \subset \{1, \ldots, N\}$ with $|J_u|, |J_v| \geq \alpha N$ and $|\langle u, \varphi_i\rangle| |\langle v, \varphi_i\rangle| \geq \frac{c^2}{M}$ for all $j \in J_u \cap J_v$. Then, since $|J_u \cap J_v| \geq (2\alpha - 1)N > 0$,

$$\sum_{i=1}^N |\langle u, \varphi_i\rangle||\langle v, \varphi_i\rangle| \geq \sum_{i \in J_u \cap J_v} |\langle u, \varphi_i\rangle||\langle v, \varphi_i\rangle| \geq \frac{c^2(2\alpha - 1)N}{M}.$$

That is, for all pairs $x, y \in \mathbb{C}^M$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq c^2(2\alpha - 1)\frac{N}{M}||x - e^{i(\theta_y - \theta_x)} y||_2 ||x + e^{i(\theta_y - \theta_x)} y||_2.$$