Frame Bounds for Gabor Frames in Finite Dimensions

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Abstract: One of the key advantages of a frame is its redundancy. Provided we have a control on the frame bounds, this redundancy allows to achieve robust reconstruction of a signal from its frame coefficients that are corrupted by noise, rounding error, or erasures. We discuss Gabor frames (g, Λ) with generic frame set Λ and random window g and show that, with high probability, such frames have frame bounds similar to the frame bounds of randomly generated frames with independent frame vectors.

Introduction
A finite set of vectors \( \Phi = \{ \varphi_j \}_{j=1}^{N} \subset \mathbb{C}^{M} \) is called a frame with frame bounds 0 < A ≤ B if, for any \( x \in \mathbb{C}^{M} \),

\[
A \|x\|_2^2 \leq \sum_{j=1}^{N} |\langle x, \varphi_j \rangle|^2 \leq B \|x\|_2^2.
\]

We identify a frame \( \Phi \) with its synthesis matrix \( \Phi \), having the frame vectors \( \varphi_i \) as its columns. Its adjoint \( \Phi^\ast \) is called the analysis matrix of the frame \( \Phi \).

The optimal lower and upper frame bounds are given by

\[
A = \min_{x \in \mathbb{C}^{M} \setminus \{0\}} \frac{1}{M} \sum_{j=1}^{N} |\langle x, \varphi_j \rangle|^2, \quad B = \max_{x \in \mathbb{C}^{M} \setminus \{0\}} \frac{1}{M} \sum_{j=1}^{N} |\langle x, \varphi_j \rangle|^2.
\]

Problem: Given noisy measurements \( c = \Phi^\ast x + \epsilon \in \mathbb{C}^{N} \), reconstruct \( x \).

We construct an estimate of \( x \) using standard dual frame:

\[
x = (\Phi^\ast)^\ast \Phi^\ast x = x + (\Phi^\ast)^\ast \Phi^\ast \epsilon.
\]

Then the reconstruction error is

\[
\|x - \hat{x}\|_2 \leq \frac{\|c - \Phi^\ast \hat{x}\|_2}{\sum_{j=1}^{N} \sigma_{j}(\Phi^\ast)}.
\]

Moreover,

\[
\frac{\|x - \hat{x}\|_2}{\|x\|_2} \leq \frac{\text{Coind}(\Phi^\ast)}{\text{SNR}},
\]

where, \( \text{Coind}(\Phi^\ast) = \frac{\text{Tr}(\Phi^\ast \Phi)}{\text{SNR}} = \sum_{j=1}^{N} \sigma_{j}(\Phi^\ast) \) is the condition number of \( \Phi^\ast \) and \( \text{SNR} = \frac{\|c\|_2^2}{\|\epsilon\|_2^2} \) is the signal to noise ratio.

Goal: Bound \( \sigma_{\max}(\Phi^\ast), \sigma_{\min}(\Phi^\ast) \) to ensure robust signal reconstruction.

Extreme singular values are sufficiently well-studied for random frames with independent entries [1], [3]. The case of structured random matrices corresponding to application relevant frames, such as Gabor frames, is not yet fully studied [2].

Definition of Gabor frames

1. Translation (or time shift) by \( k \in Z_M \), is given by \( \tau_k x = x(m - k)_{m \in Z_M} \).
2. Modulation (or frequency shift) by \( \ell \in Z_M \), is given by \( \Lambda \ell x = (2^\ell \cdot m(m - \ell))_{m \in Z_M} \).
3. The superposition \( \pi(k, \ell) = \Lambda \ell \tau_k \) of translation by \( k \) and modulation by \( \ell \) is a time-frequency shift operator.

For \( g \in \mathbb{C}^{M} \setminus \{0\} \) and \( \Lambda \subset Z_M \times Z_M \), the set of vectors \( (g, \Lambda) = \{ \pi(k, \ell) g_{(k, \ell)} \}_{(k, \ell) \in \Lambda} \) is called the Gabor system generated by the window \( g \) and the set \( \Lambda \). A Gabor system which spans \( \mathbb{C}^{M} \) is a frame and is referred to as a Gabor frame.

Main results [4]

We show that for any \( \epsilon \in (0, 1) \), a generic subframe \( (g, \Lambda) \) of \( (Z_M \times Z_M) \) with \( |\Lambda| = O(M^4 \log M) \) is a well-conditioned analysis matrix with high probability.

More precisely, for structured \( \Lambda \) we have

Proposition 1. Let \( (g, \Lambda) \) be a Gabor system with \( \Lambda = F \times Z_M, F \subset Z_M, F \neq \emptyset \), and \( g \in \mathbb{C}^{M} \). Then \( (g, \Lambda) \) is a frame if and only if \( \min_{n \in \mathbb{Z}} \|g_{(n, 0)}\| \neq 0 \), where \( g(\cdot, 0) \) is the restriction of \( g \) to the set of coefficients \( F \times \{0\} \subset Z_M \).

Moreover, in this case the optimal lower and upper frame bounds for \( (g, \Lambda) \) are

\[
A = M \min_{n \in \mathbb{Z}} \|g_{(n, 0)}\|, \quad B = M \max_{n \in \mathbb{Z}} \|g_{(n, 0)}\|,
\]

respectively.

Note: an analogous result is true for the case when \( \Lambda = Z_M \times F, \) for some \( F \subset Z_M \).

In the case of a generic frame set \( \Lambda \), we have

Theorem 2. Let \( g \) be a Steinhaus window and consider a Gabor system \( (g, \Lambda) \).

For any \( g \in Z_M \times Z_M \) and \( \epsilon \in (0, 1) \), with probability at least \( 1 - \epsilon \),

\[
\sigma_{\max}(\Phi^\ast) \leq \frac{A}{M}(1 - \epsilon) \leq \sigma_{\max}(\Phi^\ast) \leq \frac{A}{M}(1 + \epsilon).
\]

Let \( \epsilon \in (0, \frac{1}{2}) \) and \( C > 0 \) be sufficiently large constant. Let \( \Lambda \subset Z_M \times Z_M \) be a random set, constructed so that \( 1_{\Lambda(1,0)} \sim i.i.d. \text{Bernoulli}(\tau) \) for (\( k, \ell \)) \( \in Z_M \times Z_M \), where \( \tau = \frac{\sqrt{\log M}}{M} \).

Then, with high probability,

\[
\frac{A}{M}(1 - \epsilon) \leq \sigma_{\min}(\Phi^\ast) \leq \sigma_{\max}(\Phi^\ast) \leq \frac{A}{M}(1 + \epsilon).
\]

Note: the bound in Theorem 2 is tight for a full Gabor frame with \( \Lambda = Z_M \times Z_M \). In the case when \( |\Lambda| = \alpha M^2 \), for some \( \alpha \in (0, 1) \), the proven bound gives

\[
\sigma_{\max}(\Phi^\ast) \leq \frac{A}{M}\left(1 + \sqrt{\frac{25}{16} \alpha M}\right) \leq \alpha M + \frac{\sqrt{25M}}{\sqrt{16}}.
\]

This translates the problem of bounding the singular values into a combinatorial problem.

Numerical results

- Steinhaus window \( g \), that is, \( g(m) = \frac{1}{\sqrt{2\pi M}} e^{-2\pi i m/M} \) and \( y_n \sim \text{i.i.d. Unif}[0,1] \).
- \( \Lambda \) is chosen at random as described in Theorem 2, with \( \tau = \frac{\sqrt{\log M}}{M} \).

The obtained numerical results suggest that

- in the case when random \( \Lambda \) is constructed as in Theorem 2 with \( \tau = \frac{\sqrt{\log M}}{M} \), there exist constants \( 0 < \epsilon < \frac{1}{2} \) not depending on \( M \), such that all the singular values of the analysis matrix \( \Phi^\ast \) are inside the interval \( \left(\frac{\sqrt{25}}{16}, \frac{\sqrt{25}}{16}\right) \) with high probability.
- Even in the worst case scenario choice of \( \Lambda \) normalized trace expectation decreases rapidly with the dimension. This allows to conjecture that Theorem 2 can be further generalized using the proposed method.

Conclusions and Forthcoming Research

While the presented results discuss the case of a generic, randomly generated, \( \Lambda \), one of the main directions for the future research is to evaluate frame bounds of Gabor frames for all possible frame sets \( \Lambda \) and to investigate their dependencies on the structure of \( \Lambda \).

References