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Frame Bounds for Gabor Frames in Finite Dimensions

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Abstract: One of the key advantages of a frame is its redundancy. Provided we have a control on the frame bounds, this redundancy allows to achieve robust reconstruction of a signal from its frame coefficients that are corrupted by noise, rounding error, or erasures. We discuss Gabor frames (g,Λ) with generic frame set Λ and random window g and show that, with high probability, such frames have frame bounds similar to the frame bounds of randomly generated frames with independent frame vectors.

Introduction

A finite set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j\rangle|^2 \leq B||x||_2^2.$$

We identify a frame Φ with its synthesis matrix Φ, having the frame vectors φ_j as its columns. Its adjoint Φ^* is called the analysis matrix of the frame Φ.
The optimal lower and upper frame bounds are given by

$$A = \min_{x\in\mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j\rangle|^2 = \sigma_{\min}^2(\Phi^*), \quad B = \max_{x\in\mathbb{S}^{M-1}} \sum_{j=1}^N |\langle x, \varphi_j\rangle|^2 = \sigma_{\max}^2(\Phi^*).$$

Problem: Given noisy measurements $c = \Phi^* x + \delta \in \mathbb{C}^N$, reconstruct x.

We construct an estimate of x using standard dual frame:

$$\tilde{x} = (\Phi\Phi^*)^{-1}\Phi c = x + (\Phi\Phi^*)^{-1}\Phi\delta.$$

Then the reconstruction error is

$$||\tilde{x} - x||_2^2 \leq ||(\Phi\Phi^*)^{-1}\Phi||_2^2 ||\delta||_2^2 = \frac{||\delta||_2^2}{\sigma_{\min}^2(\Phi^*)}.$$

Moreover,

$$\frac{||\tilde{x} - x||_2}{||x||_2} \leq \frac{\mathrm{Cond}(\Phi^*)}{\mathrm{SNR}},$$

where, $\mathrm{Cond}(\Phi^*) = \frac{\sigma_{\max}(\Phi^*)}{\sigma_{\min}(\Phi^*)} = \frac{\sqrt{B}}{\sqrt{A}}$ is the condition number of Φ, and $\mathrm{SNR} = \frac{||\Phi^* x||_2}{||\delta||_2}$ is the signal to noise ratio.

Goal: Bound $\sigma_{\max}(\Phi^*)$, $\sigma_{\min}(\Phi^*)$ to ensure robust signal reconstruction.

Extreme singular values are sufficiently well-studied for random matrices with independent entries [1], [3]. The case of structured random matrices corresponding to application relevant frames, such as Gabor frames, is not yet fully studied [2].

Definition of Gabor frames

1. Translation (or time shift) by $k \in \mathbb{Z}_M$, is given by $T_k x = (x(m-k))_{m\in\mathbb{Z}_M}$.
2. Modulation (or frequency shift) by $\ell \in \mathbb{Z}_M$ is given by $M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m\in\mathbb{Z}_M}$.
3. The superposition $\pi(k,\ell) = M_\ell T_k$ of translation by k and modulation by ℓ is a time-frequency shift operator.

For $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the set of vectors

$$(g,\Lambda) = \{\pi(k,\ell)g\}_{(k,\ell)\in\Lambda}$$

is called the Gabor system generated by the window g and the set Λ. A Gabor system which spans $\mathbb{C}^M$ is a frame and is referred to as a Gabor frame.

Main results [4]

We show that for any $\epsilon \in (0,1)$, a generic subframe (g,Λ) of $(g, \mathbb{Z}_M\times\mathbb{Z}_M)$ with $|\Lambda| = O(M^{1+\epsilon}\log M)$ is has a well-conditioned analysis matrix with high probability.

More precisely, for structured Λ we have

Proposition 1. Let (g,Λ) be a Gabor system with $\Lambda = F\times\mathbb{Z}_M$, $F\subset\mathbb{Z}_M$, $F\neq\emptyset$, and $g\in\mathbb{C}^M$. Then (g,Λ) is a frame if and only if $\min_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2\} \neq 0$, where g_{F_m} is the restriction of g to the set of coefficients $F_m = \{m-k\}_{k\in F} \subset \mathbb{Z}_M$.
Moreover, in this case the optimal lower and upper frame bounds for (g,Λ) are $A = M\min_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2^2\}$ and $B = M\max_{m\in\mathbb{Z}_M}\{||g_{F_m}||_2^2\}$, respectively.

Note: an analogous result is true for the the case when $\Lambda = \mathbb{Z}_M \times F$, for some $F\subset\mathbb{Z}_M$.

In the case of a generic frame set Λ, we have

Theorem 2. Let g be a Steinhaus window and consider a Gabor system (g,Λ).

1. For any $\Lambda\subset\mathbb{Z}_M\times\mathbb{Z}_M$ and $\varepsilon\in(0,1)$, with probability at least $1-\varepsilon$,
$$\sigma_{\max}^2(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M} + \sqrt{\frac{|\Lambda|}{\varepsilon}\left(1 - \frac{|\Lambda|}{M^2}\right)}.$$
2. Let $\varepsilon\in(0,\frac{1}{2}]$ and $C>0$ a sufficiently large constant. Let $\Lambda\subset\mathbb{Z}_M\times\mathbb{Z}_M$ be a random set, constructed so that $\mathbf{1}_{\{(k,\ell)\in\Lambda\}} \sim$ i.i.d. Bernoulli(τ) for all $(k,\ell)\in\mathbb{Z}_M\times\mathbb{Z}_M$, where $\tau = \frac{C\log M}{M^{1-\varepsilon}}$. Then, with high probability,
$$\frac{|\Lambda|}{M}(1-\delta) \leq \sigma_{\min}^2(\Phi_\Lambda^*) \leq \sigma_{\max}^2(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M}(1+\delta).$$

Note: the bound in Theorem 2.1 is tight for a full Gabor frame with $\Lambda = \mathbb{Z}_M\times\mathbb{Z}_M$. In the case when $|\Lambda| = \alpha M^2$, for some $\alpha\in(0,1)$, the proven bound gives $\sigma_{\max}^2(\Phi_\Lambda^*) \leq \left(\alpha + \sqrt{\frac{\alpha(1-\alpha)}{\varepsilon}}\right) M$ with probability at least $1-\varepsilon$, which is similar to the bound for matrices with independent entries obtained in [1].

Idea of the proof

Let Λ be a general subset of $\mathbb{Z}_M\times\mathbb{Z}_M$. Fix any $m\in\mathbb{N}$ and consider the matrix $H = \Phi_\Lambda\Phi_\Lambda^* - \frac{|\Lambda|}{M} I_M$. Following the idea of [2], we obtain that

$$\mathbb{P}\left\{\frac{|\Lambda|}{M}(1-\delta) \leq \sigma_{\min}^2(\Phi_\Lambda^*) \leq \sigma_{\max}^2(\Phi_\Lambda^*) \leq \frac{|\Lambda|}{M}(1+\delta)\right\} = \mathbb{P}\left\{||H||_2 \leq \frac{|\Lambda|}{M}\delta\right\}$$
$$= \mathbb{P}\left\{||H||_2^{2m} > \frac{|\Lambda|^{2m}}{M^{2m}}\delta^{2m}\right\} \leq \frac{M^{2m}}{|\Lambda|^{2m}}\delta^{-2m}\mathbb{E}(||H^m||_2^2) \leq \frac{M^{2m}}{|\Lambda|^{2m}}\delta^{-2m}\mathbb{E}(\mathrm{Tr}\, H^{2m}).$$

One can further show that [4]

$$\mathbb{E}(\mathrm{Tr}\, H^m) = \sum_{\substack{j_1,j_2,\dots,j_m\in\mathbb{Z}_M,\\ j_1\neq j_2\neq\cdots\neq j_m\neq j_1\\ (k_1,\ell_1),\dots,(k_m,\ell_m)\in\Lambda}} e^{\frac{2\pi i}{M}\sum_{t=1}^m \ell_t(j_t - j_{t+1})} E_{\substack{j_1\dots j_m\\ k_1\dots k_m}},$$

where

$$E_{\substack{j_1\dots j_m\\ k_1\dots k_m}} = \begin{cases} \frac{1}{M^m}, & \text{if } \exists\alpha\in S_m, \text{ s.t. } j_t - k_t = j_{\alpha(t)} - k_{\alpha(t)-1},\ \forall t\in\{1,\dots,m\}; \\ 0, & \text{otherwise.}\end{cases}$$

This translates the problem of bounding the singular values into a combinatorial problem.

Numerical results

- Steinhaus window g, that is, $g(m) = \frac{1}{\sqrt{M}}e^{2\pi i y_m}$ and $y_m \sim$ i.i.d. Unif$[0,1)$.
- Λ is chosen at random as described in Theorem 2, with $\tau = \frac{C}{M}$.

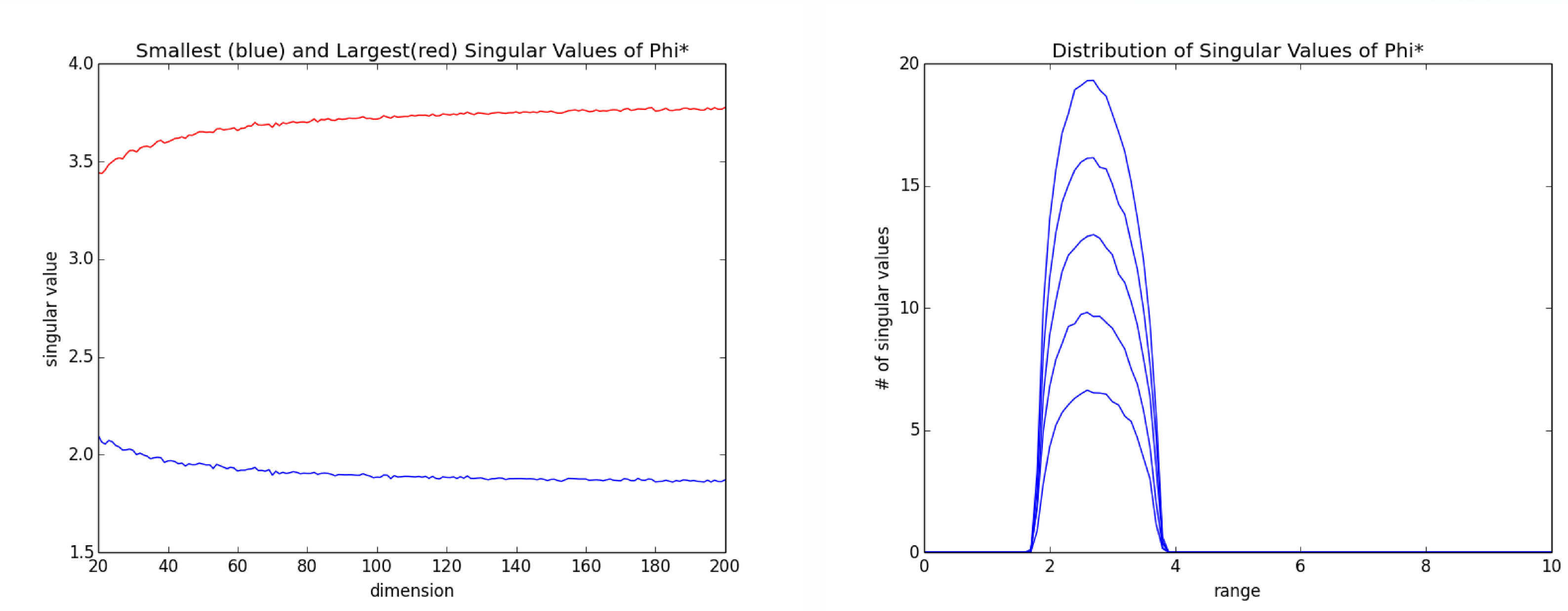

Figure 1 – **Left:** the dependence of the extreme singular values of the analysis matrix Φ_Λ^* of a Gabor frame (g,Λ) on the ambient dimension M; **Right:** the distribution of the singular values of Φ_Λ^* for the dimensions $M = 100, 150, 200, 250, 300$.

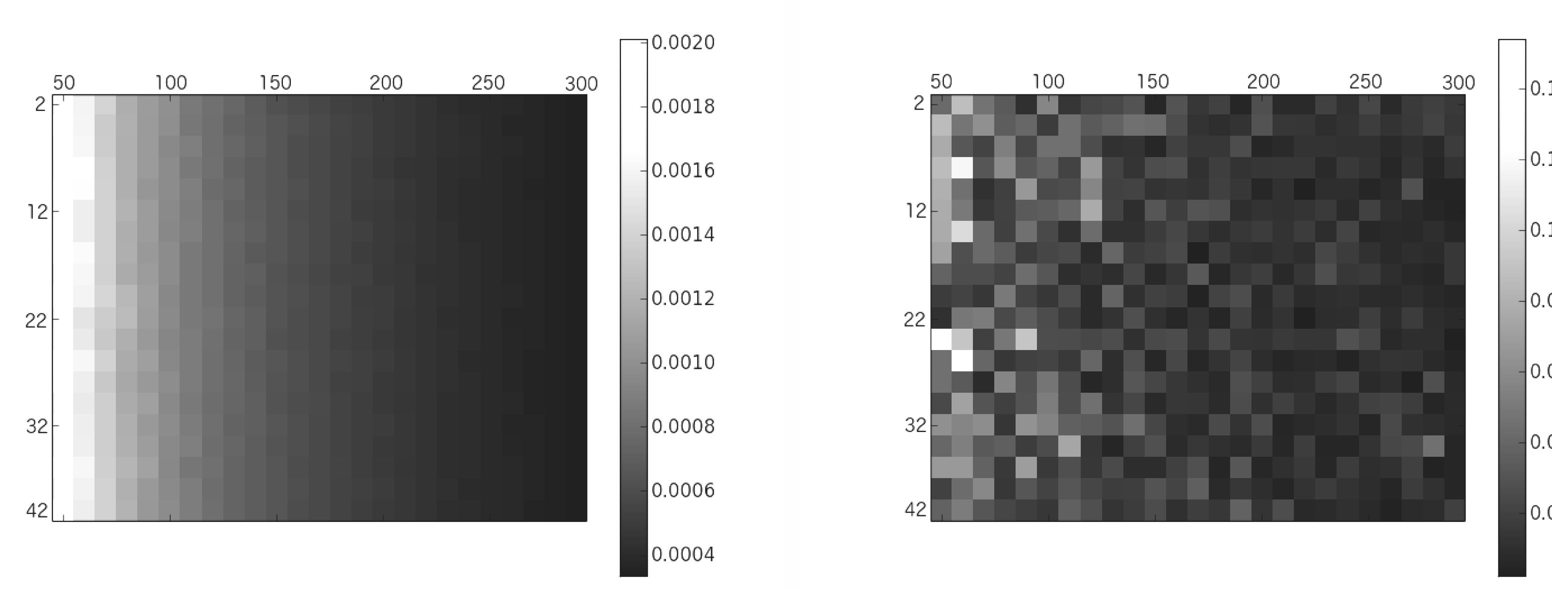

Figure 2 – The dependence of the numerically estimated trace expectation $\frac{M^{2m}}{|\Lambda|^{2m}}\delta^{-2m}\mathbb{E}(\mathrm{Tr}\, H^{2m})$ on the ambient dimension M (horizontal axis) and the parameter C (vertical axis), for a fixed m. **Left:** Λ is chosen at random, as described in Theorem 2, with $\tau = \frac{C}{M}$; **Right:** $\Lambda = F\times\{0,1,\dots,\lfloor\frac{M}{2}\rfloor\}$.

The obtained numerical results suggest that

1. In the case when random Λ is constructed as in Theorem 2 with $\tau = \frac{C}{M}$, there exist constants $0<k<K$ not depending on M, such that all the singular values of the analysis matrix Φ_Λ^* are inside the interval $\left[k\frac{|\Lambda|}{M}, K\frac{|\Lambda|}{M}\right]$ with high probability.
2. Even in the worst case scenario choice of Λ normalized trace expectation decreases rapidly with the dimension. This allows to conjecture that Theorem 2 can be further generalized using the proposed method.

Conclusions and Forthcoming Research

While the presented results discuss the case of a generic, randomly generated, Λ, one of the main directions for the future research is to evaluate frame bounds of Gabor frames for all possible frame sets Λ and to investigate their dependencies on the structure of Λ.

References

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